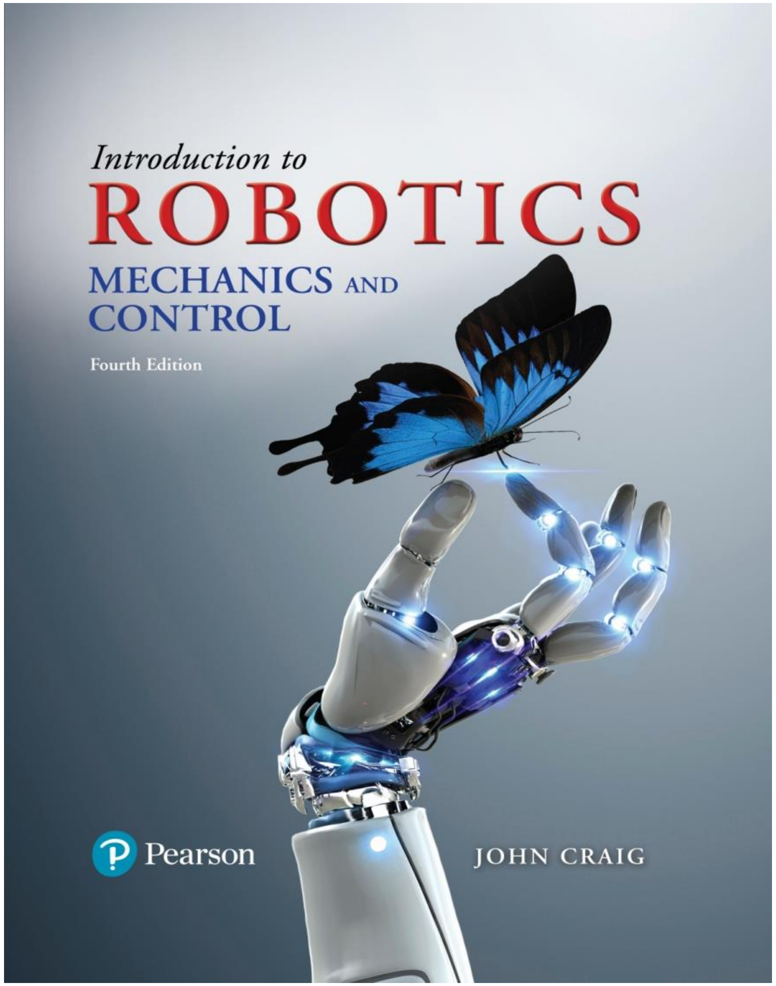
Introduction to Robotics

Mechanics and Control

4th Edition



Chapter 06

Manipulator Dynamics

Pearson

6.1 Introduction

To this point, we have focused on kinematics of manipulators – static positions, static forces and velocities. Now we will consider the equations of motion of a manipulator- the way the manipulator moves due to torques and external forces.

Two problems related to manipulator dynamics will be addressed.

1) Given a trajectory – meaning θ, $\dot{\theta}$, $\ddot{\theta}$ - find the required joint torques.

2) Given the joint torques – calculate the resulting motion – find θ, $\dot{\theta}$, $\ddot{\theta}$

6.2 Acceleration of a Rigid Body

The rigid body linear and angular velocities have derivatives that are the linear and angular accelerations.

$$^{B}\dot{V}_{Q} = \frac{d}{dt}\,^{B}V_{Q} = \lim_{\Delta t \to o} \frac{^{B}V_{Q}(t+\Delta t) - ^{B}V_{Q}(t)}{\Delta t}$$

$$^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,^{A}\Omega_{B} = \lim_{\Delta t \to o} \frac{^{A}\Omega_{B}(t+\Delta t) - ^{A}\Omega_{B}(t)}{\Delta t}$$

When accelerations are defined in universal reference frames we will use the notation

$$\dot{\upsilon}_{A} = ^{U}\dot{V}_{AORG}$$

$$\dot{\omega}_{A} = ^{U}\dot{\Omega}_{A}.$$

Pearson

6.2 Acceleration of a Rigid Body

Linear Acceleration

Recall the velocity ${}^{B}Q$ as seen from $\{A\}$ when the origins are coincident.

$$ {}^{A}V_{Q} = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q $$

We can write this as

$$ \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q. \quad ** $$

Differentiating, we get the acceleration of ${}^{B}Q$ with respect to $\{A\}$ when the origins of $\{A\}$ and $\{B\}$ coincide.

$$ {}^{A}\dot{V}_{Q} = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}V_{Q}) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) $$

Now apply the derivative to both the first and last terms

$$ \begin{aligned} {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} &+ {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q \\ &+ {}^{A}\Omega_{B} \times ({}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q). \end{aligned} $$

Pearson

6.2 Acceleration of a Rigid Body

Linear Acceleration

Combining the terms we get

$$ {}^A_BR\,{}^B\dot{V}_Q + 2\,{}^A\Omega_B \times {}^A_BR\,{}^BV_Q + {}^A\dot{\Omega}_B \times {}^A_BR\,{}^BQ + {}^A\Omega_B \times ({}^A\Omega_B \times {}^A_BR\,{}^BQ). $$

To generalize to the case where the origins are not coincident, we add the linear acceleration of the origin of {*B*}

$$ {}^A\dot{V}_{BORG} + {}^A_BR\,{}^B\dot{V}_Q + 2\,{}^A\Omega_B \times {}^A_BR\,{}^BV_Q + {}^A\dot{\Omega}_B \times {}^A_BR\,{}^BQ \qquad *** $$
$$ + {}^A\Omega_B \times ({}^A\Omega_B \times {}^A_BR\,{}^BQ). $$

In the case where ${}^BQ$ is constant or $\quad {}^BV_Q = {}^B\dot{V}_Q = 0$

The acceleration in {A} reduces to

$$ {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A\Omega_B \times ({}^A\Omega_B \times {}^A_BR\,{}^BQ) + {}^A\dot{\Omega}_B \times {}^A_BR\,{}^BQ. $$

This will be used when calculating the linear acceleration of links with rotational joints. Eqn *** will be used for links with prismatic joints.

6.2 Acceleration of a Rigid Body

Angular Acceleration

In cases where we have {*B*} rotating relative to {*A*} with ${}^{A}\Omega_{B}$ and {C} is rotating relative to {*B*} with ${}^{B}\Omega_{C}$ To find ${}^{A}\Omega_{C}$

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\, {}^{B}\Omega_{C} $$

Taking the derivative yields

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}({}^{A}_{B}R\, {}^{B}\Omega_{C}) $$

Using

$$ \frac{d}{dt}({}^{A}_{B}R\, {}^{B}Q) = {}^{A}_{B}R\, {}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}Q. $$

we get

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + {}^{A}_{B}R\, {}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}\Omega_{C}. $$

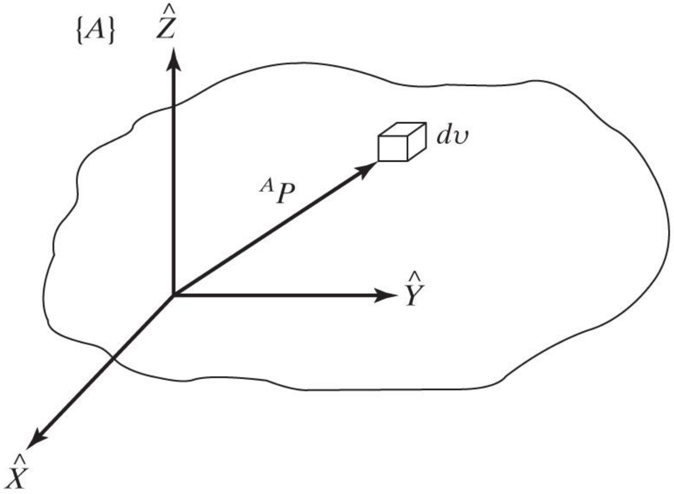
This will be used to calculate the angular acceleration of the links.

6.3 Mass Distribution

For single DOF systems, we often refer to the mass of a rigid body. For rotational motion about a single axis we refer to the moment of a inertia. For a rigid body that can move in three dimensions, there are infinitely many rotation axes. For an arbitrary body we need a general way to characterize the mass distribution of a rigid body. We then define the inertia tensor.

The figure shows a rigid body with an attached frame {A}. (can use any frame)

The inertia tensor relative to frame {A} is expressed using a 3 x 3 matrix



The inertia tensor of an object describes the object's mass distribution. Here, the vector ^{A}P locates the differential volume element, $d\upsilon$.

6.3 Mass Distribution

$$
{}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

where the scaler elements are

$$I_{xx} = \iiint_V (y^2 + z^2)\rho dv,$$

$$I_{yy} = \iiint_V (x^2 + z^2)\rho dv,$$

$$I_{zz} = \iiint_V (x^2 + y^2)\rho dv,$$

$$I_{xy} = \iiint_V xy\rho dv,$$

$$I_{xz} = \iiint_V xz\rho dv,$$

$$I_{yz} = \iiint_V yz\rho dv,$$

The rigid body is composed of differential volume elements *dv* with density ρ. *Elements are located with* ${}^{A}P = [\ x\ y\ z]^T$.

I_{xx} I_{yy} and I_{zz} are the mass moments of inertia. The elements with mixed indices are the mass products of inertia. The set of six quantities will depend of the position and orientation of the frame in which they are described.

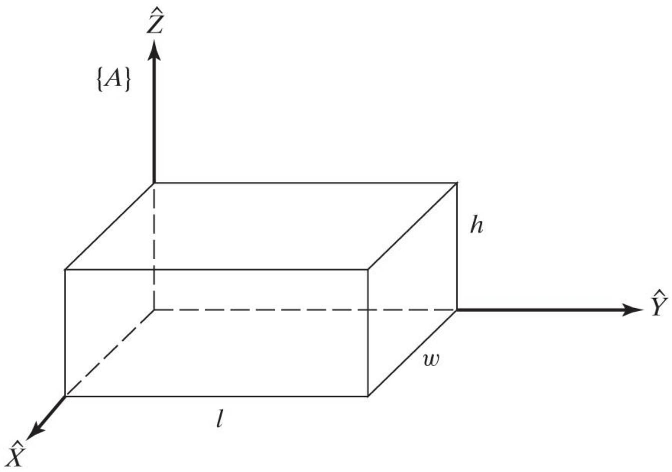
Can choose frame orientation so the mass products are zero – defines principal axes.

6.3 Mass Distribution

Example 6.1

Find the inertia tensor for the rectangular body of density ρ with respect to the axis shown in the figure.
First we calculate I_{xx} using the volume element $dv = dx\ dy\ dz$

$$I_{xx} = \int_0^h \int_0^l \int_0^\omega (y^2 + z^2)\rho\, dx\, dy\, dz$$
$$= \int_0^h \int_0^l (y^2 + z^2)\omega\rho\, dy\, dz$$
$$= \int_0^h \left(\frac{l^3}{3} + z^2 l\right)\omega\rho\, dz$$
$$= \left(\frac{hl^3\omega}{3} + \frac{h^3 l\omega}{3}\right)\rho$$
$$= \frac{m}{3}(l^2 + h^2),$$



A body of uniform density.

6.3 Mass Distribution

Example 6.1

Similarly we get I_{yy} and I_{zz}

$$I_{yy} = \frac{m}{3}(\omega^2 + h^2)$$

$$I_{zz} = \frac{m}{3}(l^2 + \omega^2).$$

Then we calculate I_{xy}

$$\begin{aligned} I_{xy} &= \int_0^h \int_0^l \int_0^\omega xy\rho \, dx \, dy \, dz \\ &= \int_0^h \int_0^l \frac{\omega^2}{2} y\rho \, dy \, dz \\ &= \int_0^h \frac{\omega^2 l^2}{4} \rho dz \\ &= \frac{m}{4}\omega l. \end{aligned}$$

and

$$I_{xz} = \frac{m}{4}h\omega$$

$$I_{yz} = \frac{m}{4}hl.$$

Finally – the inertia tensor is

$${}^{A}I = \begin{bmatrix} \frac{m}{3}(l^2 + h^2) & -\frac{m}{4}\omega l & -\frac{m}{4}h\omega \\ -\frac{m}{4}\omega l & \frac{m}{3}(\omega^2 + h^2) & -\frac{m}{4}hl \\ -\frac{m}{4}h\omega & -\frac{m}{4}hl & \frac{m}{3}(l^2 + \omega^2) \end{bmatrix}$$

Pearson

6.3 Mass Distribution

Using the parallel axis theorem, we can describe the inertia tensor by translating the reference coordinate system

$$^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2),$$

$$^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c,$$

where $\{C\}$ is located at center of mass and x_c, y_c and z_c locates the center of mass with respect to $\{A\}$. In matrix form this can be written as

$$^{A}I = {}^{C}I + m[P_c^T P_c I_3 - P_c P_c^T],$$

Example 6.2

Find the inertia tensor for the previous rectangle with the origin at the body's center of mass.

We can apply the parallel axis theorem where

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \omega \\ l \\ h \end{bmatrix}$$

6.3 Mass Distribution

Example 6.2

Then we get

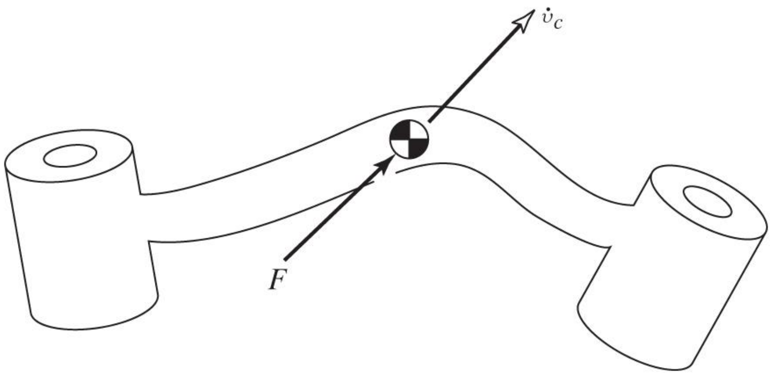
$$
\begin{aligned}
{}^{C}I_{zz} &= \frac{m}{12}(\omega^2 + l^2) \\
{}^{C}I_{xy} &= 0.
\end{aligned}
$$

and the other are elements are found by symmetry

$$
{}^{C}I = \begin{bmatrix} \frac{m}{12}(h^2 + l^2) & 0 & 0 \\ 0 & \frac{m}{12}(\omega^2 + h^2) & 0 \\ 0 & 0 & \frac{m}{12}(l^2 + \omega^2) \end{bmatrix}
$$

6.4 Newton's Equation, Euler's Equation

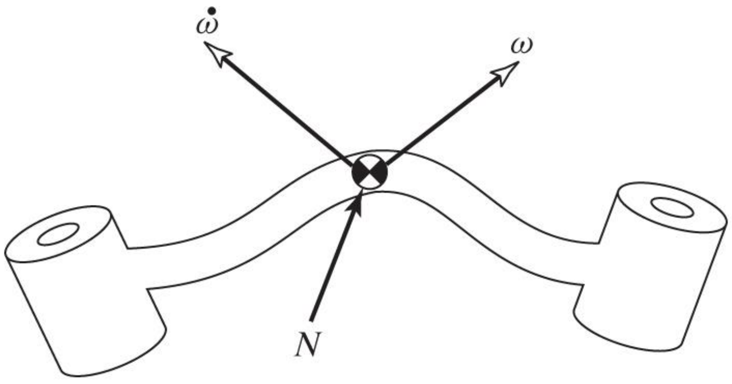
In order to describe the motion of each link of a manipulator we will use Newton's equation (linear) and Euler's equation (rotation) to describe how the forces, inertias and accelerations relate.



A force *F* acting at the center of mass of a body causes the body to accelerate at υ_C.

$$F = m\dot{v}_C$$

where m is the total mass of the link.



A moment *N* is acting on a body, and the body is rotating with velocity ω and accelerating at ω.

$$N = {}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega.$$

${}^{C}I$ is the inertia tensor in $\{C\}$ located at the center of mass

6.5 Iterative Newton-Euler Dynamic Formulation

In this section we determine the required torques for given position, velocity and acceleration of the joints (θ, $\dot{\theta}$, $\ddot{\theta}$). With these values, information on the kinematics and mass distribution, we can determine the joint torques that cause the motion.

Outward Iterations to Compute Velocities and Accelerations

The propagation of the rotational velocity was discussed in Ch 5 and is shown here as

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$

From the previously derived

$$^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}_{B}R\, {}^{B}\dot{\Omega}_C + {}^{A}\Omega_B \times {}^{A}_{B}R\, {}^{B}\Omega_C.$$

we can write the angular acceleration from one link to the next as

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$

If the joint is prismatic then the equation reduces to

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i.$$

6.5 Iterative Newton-Euler Dynamic Formulation

For the linear velocity we found

$$^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BORG} + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}Q) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\, {}^{B}Q.$$

Similarly we get the following for the linear acceleration

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_{i}]$$

For a prismatic link this equation becomes

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_{i})$$
$$+ 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}.$$

We will also need the linear acceleration of the center of mass of each link (applying the top equation).

$$^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_{i} \times {}^{i}P_{C_i} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} + {}^{i}P_{C_i}) + {}^{i}\dot{v}_{i}.$$

The Force and Torque Acting on a Link

To get the inertial force and torque acting on each link we use the Newton-Euler equations.

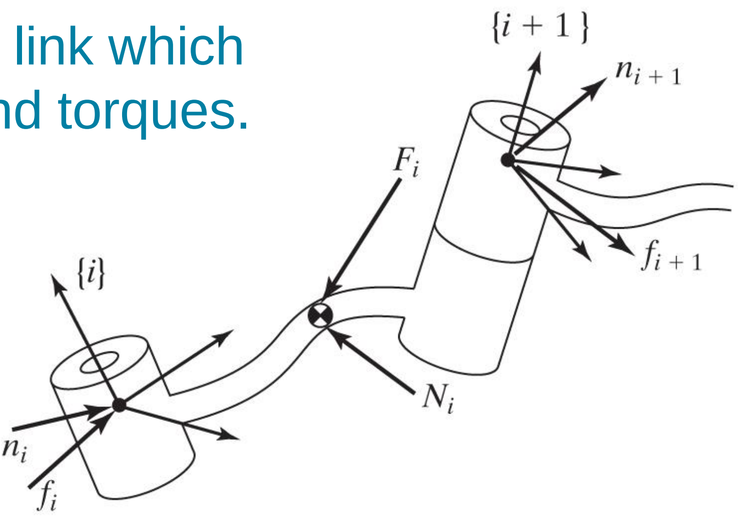
$$F_i = m\dot{v}_{C_i},$$

$$N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i.$$

6.5 Iterative Newton-Euler Dynamic Formulation

Inward Iterations to Compute Forces and Torques

The figure shows a free body diagram of a link which can be used to determine the net forces and torques.
f_i = force exerted on link i by link i-1
n_i = torque exerted on link i by link i-1



The force balance, including inertial forces, for a single manipulator link.

Summing forces and moments about the center of mass yields

$$^{i}F_i = {}^{i}f_i - {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}$$

$$^{i}N_i = {}^{i}n_i - {}^{i}n_{i+1} + (-{}^{i}P_{C_i}) \times {}^{i}f_i - ({}^{i}P_{i+1} - {}^{i}P_{C_i}) \times {}^{i}f_{i+1}$$

Using the results from the force balance and adding rotation matrices we get

$$^{i}N_i = {}^{i}n_i - {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} - {}^{i}P_{C_i} \times {}^{i}F_i - {}^{i}P_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}$$

6.5 Iterative Newton-Euler Dynamic Formulation

Now we can rearrange the order of the two dynamic equations – and order the propagation from the higher link to lower link.

$$^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i,$$

$$^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}$$

These are the inward force iterations.

The required joint torques are found by taking the Z component of the torque applied by one link to the next

$$\tau_i = {}^{i}n_i^T\, {}^{i}\hat{Z}_i.$$

If the joint is prismatic then we use

$$\tau_i = {}^{i}f_i^T\, {}^{i}\hat{Z}_i.$$

where τ is now the linear actuator force

Pearson

6.5 Iterative Newton-Euler Dynamic Formulation

The Iterative Newton-Euler Dynamics Algorithm

The complete algorithm for calculating joint torques from the motion of the torques consists of 2 parts – 1) link velocities and accelerations are found from link 1 to n and the Newton-Euler equations are applied to each link 2) forces and torques of interaction and joint actuator torques are calculated from link n back to 1. A summary for rotational joints is shown below.

Outward iterations: $i : 0 \rightarrow 5$

$$
\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i), \\
{}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\
&\quad + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}, \\
{}^{i+1}F_{i+1} &= m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}, \\
{}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}.
\end{aligned}
$$

6.5 Iterative Newton-Euler Dynamic Formulation

The Iterative Newton-Euler Dynamics Algorithm

Inward iterations: $i : 6 \rightarrow 1$

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} + {}^{i}F_i, \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\
&\quad + {}^{i}P_{i+1} \times {}^{i}_{i+1}R \, {}^{i+1}f_{i+1}, \\
\tau_i &= {}^{i}n_i^{T} \, {}^{i}\hat{Z}_i.
\end{aligned}
$$